III. Quillen's superconnection

Goal: $\mathrm{Ind}(P)$ analytical index of elliptic op can be treated as a "characteristic form" via Quillen's superconnection

III.1 Superconnection on superbundle

Def: A superbundle E on X is $E = E^+ \oplus E^-$ where E^+, E^- are two vector bundles on X s.t. $\forall x \in X$ $E_x = E_x^+ \oplus E_x^-$ is a super vector space.

For $E = E^+ \oplus E^-$

$$\Omega^\bullet(X, E) = C^\infty(X, \Lambda^\bullet T^*X \hat{\otimes} E)$$

$\mathbb{Z}/2\mathbb{Z}$-graded or super-space

Def: A $\mathbb{C}$-linear map

$$A : \Omega^\bullet(X, E) \to \Omega^\bullet(X, E)$$

is called a super connection if

① A is an odd operator (exchange the $\mathbb{Z}/2\mathbb{Z}$-grading on $\Omega^\bullet(X,E)$)

② $A(\alpha \wedge s) = d\alpha \wedge s + (-1)^{|\alpha|} \alpha \wedge As$

$\alpha \in \Omega^\bullet(X)$, $s \in \Omega^\bullet(X, E)$

Rmk : An equivalent definition of a superconnection A

$$A : \Gamma(E) \to \Omega^{\bullet}(X, E)$$

is a first order different op, which is odd parity and $\sigma(A)(x,\xi) = \sqrt{-1}\,\xi\wedge$. Then for $\alpha \in \Omega^{\bullet}(X)$, $s \in \Omega^{\bullet}(X, E)$

Take $A(\alpha\wedge s) = d\alpha\wedge s + (-1)^{|\alpha|}\alpha\wedge As$.

Prop : $$A = A_0 + \nabla^E + A_2 + A_3 + \cdots + A_m$$

where $A_{2j} \in \Omega^{2j}(X, \mathrm{End}^-(E))$

$A_{2j+1} \in \Omega^{2j+1}(X, \mathrm{End}^+(E))$

$\nabla^E = \nabla^{E^+} \oplus \nabla^{E^-}$ a classical connection.

Def : $F = A^2$ curvature of A

Rmk : $$A^2(fs) = A(df\, s + f As)$$

$$= \underbrace{-df\wedge As + df\wedge As}_{0} + f A^2 s$$

so $A^2 \in \Omega^{even}(X, \mathrm{End}^+(E)) \oplus \Omega^{odd}(X, \mathrm{End}^-(E))$

Prop (Bianchi) $[A, F] = 0$.

Def : ① Number operator $N : \Omega^{\bullet}(X) \to \Omega^{\bullet}(X)$

$N|_{\Omega^k(X)} = k\,\mathrm{Id}$

② Normalization map

$$\varphi : \Omega^{even}(X) \to \Omega^{even}(X)$$
$$\alpha \mapsto \frac{1}{(2\pi i)^{N/2}} \alpha$$

Number op.

Theorem: $E = E^+ \oplus E^-$ a complex superbundle on X

$\mathbb{A}$ a superconnection

$f(x) = \sum_j a_j x^j$ a formal power series

Then

① $\varphi \operatorname{Tr}_s[f(-\mathbb{A}^2)] \in \Omega^{even}(X)$ is a closed form.

② Its cohomological class does not depend on the choice of $\mathbb{A}$

Moreover

$$[\varphi \operatorname{Tr}_s[f(-\mathbb{A}^2)]]_{dR} = f_a(E^+) - f_a(E^-) \in H^{\cdot}(X).$$

Proof: ① Since $\mathbb{A} = \mathbb{A}_0 + \nabla^E + \mathbb{A}_2 + \cdots$

connection

$$\Rightarrow d \operatorname{Tr}_s[\alpha] = \operatorname{Tr}_s[[\mathbb{A}, \alpha]]$$

Therefore $d \operatorname{Tr}_s[f(-\mathbb{A}^2)] = \operatorname{Tr}_s[[\mathbb{A}, f(-\mathbb{A}^2)]]$

$$\underset{\text{Bianchi}}{=} 0$$

② Independence of $\mathbb{A}$ is similar to the case of $f_a(E, \nabla^E)$.

HW 3.1

$$\nabla^E = 0 + \underset{\nabla^{E^+} \oplus \nabla^{E^-}}{\underset{\shortparallel}{\nabla^E}} + 0 + \cdots \quad \text{a superconnection}$$

④

Then

$$\varphi \operatorname{Tr}_s[f(-A^2)] \overset{\text{cohom.}}{=\!=} \varphi \operatorname{Tr}_s[f(-R^{E^+} \oplus R^{E^-})]$$

$$= \varphi \operatorname{Tr}_s\left[\begin{bmatrix} f(-R^{E^+}) & \\ & f(-R^{E^-}) \end{bmatrix}\right]$$

$$= \operatorname{Tr}^{E^+}[f(\tfrac{\sqrt{-1}}{2\pi}R^{E^+})] - \operatorname{Tr}^{E^-}[f(\tfrac{\sqrt{-1}}{2\pi}R^{E^-})]$$

$$= f_a(E^+) - f_a(E^-)$$

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Rk: $f(x) = e^x \rightsquigarrow$ chern character

III.2 The baby case $X = pt$.

vector bundle = vector space

$E = E^+ \oplus E^-$

$D : E^+ \to E^-$ a linear operator

Take Hermitian metrics h^{E^+}, h^{E^-} on E^+, E^-

Def: $D^* : E^- \longrightarrow E^+$ adj op. of D w.r.t. $h^{E^\pm}$

is the unique linear op. s.t. $\forall\ a \in E^+,\ b \in E^-$

$$h^{E^+}(D^*b, a) = h^{E^-}(b, Da)$$

Now take

$$A = \begin{bmatrix} 0 & D^* \\ D & 0 \end{bmatrix} \in \operatorname{End}^-(E)$$

but it is self-adjoint

is a superconnection

Then

$$A^2 = \begin{bmatrix} D^*D & 0 \\ 0 & DD^* \end{bmatrix}$$

Now chern character

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$$ch(E, A) = Tr_s[e^{-A^2}] \in \Omega^0(pt) = \mathbb{C}$$

$$\| \quad H^0(pt)$$

$$= Tr_s\left[e^{-\begin{bmatrix} D^*D & 0 \\ 0 & DD^* \end{bmatrix}}\right]$$

Replace A by $\sqrt{t}A$, still superconnection

$\Rightarrow ch(E, \sqrt{t}A)$ is independent of $t > 0$

HW 3.2

Then

$$ch(E, A) = \lim_{t \to +\infty} ch(E, \sqrt{t}A)$$

$D^*D \in End(E^+)$

$$= \lim_{t \to +\infty} Tr_s\left[e^{-t\begin{bmatrix} D^*D & 0 \\ 0 & DD^* \end{bmatrix}}\right]$$

$\cong (Im\, D)^\perp$

$D^*D \geq 0$

$$= \dim \ker D - \dim \ker D^*$$

$$= \dim \ker D - \dim \operatorname{coker} D = Ind(D)$$

" operator is superconnection and $Ind(D)$ is a chern character "

Rmk: $\lim_{t \to 0} ch(E, \sqrt{t}A) = Tr_s^E[Id] = \dim E^+ - \dim E^-$

$= ch(E^+) - ch(E^-)$

classical chern character

Index theorem for dim = 0

$$Ind(D) = \int_{pt} ch(E^+) - ch(E^-)$$

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III.3 Chern character $ch(\sigma(P))$

P elliptic op $P(E) \to P(F)$

polynomials along T^*X

$\Rightarrow \quad \sigma(P) \in C^\infty(X, S(TX) \otimes Hom(E, F))$

$\pi : T^*X \to X$ $\quad \subset C^\infty(T^*X, Hom(\pi^*E, \pi^*F))$

Now : we regard $\underset{+}{\pi^*E} \oplus \underset{-}{\pi^*F}$ as a superbundle on T^*X

We also fix Hermitian metrics $g^E, g^F \rightsquigarrow$ on π^*E, π^*F

$$\begin{array}{c} \pi^*E, \pi^*F \\ \downarrow \\ T^*X \end{array}$$

$\sigma(P)_{(x,\xi)} \in Hom(\pi^*E, \pi^*F)$

$\rightsquigarrow$ pointwise adj $\quad \sigma(P)^*_{(x,\xi)} \in Hom(\pi^*F, \pi^*E)$

$$A = \begin{bmatrix} 0 & \sigma(P)^* \\ \sigma(P) & 0 \end{bmatrix} + \nabla^{\pi^*(E\oplus F)}$$

a superconnection on $\pi^*(E\oplus F) \to T^*X$

Definition : $ch(\sigma(P)) = \varphi \, Tr_s[\exp(-A^2)] \in \Omega^{even}(T^*X)$

Rk : $$A^2 = \begin{bmatrix} \sigma(P)^*\sigma(P) & 0 \\ 0 & \sigma(P)\,\sigma(P)^* \end{bmatrix} + \text{nilpotent terms}$$

if $\deg P = k > 0$

$A^2 \sim$ grows like $|\xi|^{2k}$ as $|\xi| \to +\infty$

As a consequence, coefficients of $ch(\sigma(P))$ decays $\sim e^{-|\xi|^2}$ on T^*X when X is compact, $\forall \alpha \in \Omega^{\cdot}(X)$,

$$\int_{T^*X} ch(\sigma(P)) \wedge \pi^*\alpha \quad \text{is always well-defined}$$

<u>In fact</u>: $[ch(\sigma(P))] \in H_c^{even}(T^*X)$, independent of metrics, connections or the smooth deformations $\{P_t\}_{t\in\mathbb{R}}$

$\uparrow$ cpt suppt

<u>Rmk</u>:
- Superconnection theory is not needed for the proof of ~~Atiyah~~-Singer index ~~theory~~, but helps us to understand ~~it~~.
- It is a good approach for the family index theory.

Rmk & Questions: $P: C^\infty(X,E) \to C^\infty(X,F)$ elliptic

1) $Tr_s\left[\exp\left(-t\begin{bmatrix} 0 & P^* \\ P & 0\end{bmatrix}^2\right)\right]$ well-defined? Yes

infinite-dimensional supertrace of heat operator on X.

2) // is independent of $t>0$? Yes

3) $\lim_{t\to+\infty} Tr_s[\cdots] = Ind(P)$? Yes

4) $\lim_{t\to 0^+} Tr_s[\cdots] = \int_{T^*X} \cdots$? This local limite exists only for $P =$ Dirac op!

"heat kernel approach for index theory of Dirac op"